The Power of Generating Functions

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Generating functions is a very powerful way to find closed formula for sequences defined iteratively.

I was so bored during the final week, so I went on internet for fun. Finally I found someone from Sydney University was asking for help on this question:

(a) If $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ and $L_0 = 2$, $L_1 = 1$, please Find a closed formula for L_n.

(b) $S_n = L_n + L_{n-1} + \ldots + L_0$, please find an closed formula for S_n.

(c) $A_n = L_{n-1}S_0 + L_{n-2}S_1 + \ldots + L_0 S_{n-1}$, please find a closed formula for A_n.

It's pretty easy to use generating functions to solve this problem.

(a) Define $L(x) = \sum_{n=0}^{\infty} L_n x^n$. From $L_n = L_{n-1} + L_{n-2}, n \geq 2$ we have

$$L_n x^n = L_{n-1}x^n + L_{n-2}x^n,\ n \geq 2$$

Sum both side we have

$$\sum_{n\geq 2} L_n x^n = x\sum_{n\geq 2} L_{n-1}x^{n-1} + x^2 \sum_{n\geq 2} L_{n-2}x^{n-2}$$

i.e.

$$L(x) - L_1 x - L_0 = x(L(x) - L_0) + x^2 L(x)$$

So we have

$$L(x) = \frac{(L_0 - L_1)x - L_0}{x^2 + x - 1} = \frac{x-2}{x^2+x-1} = \frac{-\omega_1}{x-\omega_1} + \frac{-\omega_2}{x-\omega_2} = \frac{1}{1-\frac{x}{\omega_1}} + \frac{1}{1-\frac{x}{\omega_2}}$$

where $\omega_1 = \frac{-1-\sqrt{5}}{2}$ and $\omega_2 = \frac{-1+\sqrt{5}}{2}$. Using Taylor expansion we have

$$L(x) = \sum_{n\geq 0}\left(\frac{x}{\omega_1}\right)^n + \sum_{n\geq 0}\left(\frac{x}{\omega_2}\right)^n = \sum_{n\geq 0}\left(\left(\frac{1}{\omega_1}\right)^n + \left(\frac{1}{\omega_2}\right)^n\right)x^n$$

So we know that

$$L_n = \left(\frac{1}{\omega_1}\right)^n + \left(\frac{1}{\omega_2}\right)^n = \left(\frac{-\sqrt{5}+1}{2}\right)^n + \left(\frac{\sqrt{5}+1}{2}\right)^n,\ n \geq 0.$$

(b) Define $S(x) = \sum_{n\geq 0} S_n x^n$. From $S_n = L_n + \ldots + L_0, n \geq 0$ we have

$$S_n x^n = \sum_{i=0}^{n} L_i x^n,\ n \geq 0$$

Sum both sides we have

$$\sum_{n\geq 0} S_n x^n = \sum_{n\geq 0}\sum_{i=0}^{n} L_i x^n = \sum_{i\geq 0}\sum_{n=i}^{\infty} L_i x^n = \sum_{i\geq 0} L_i \frac{x^i}{1-x} = \frac{1}{1-x}\sum_{i\geq 0} L_i x^i = \frac{L(x)}{1-x}$$

i.e.

$$S(x) = \frac{L(x)}{1-x}$$

Then similar to what we have done in part (a), we can write $S(x)$ as $S(x) = \frac{A}{x-\omega_1} + \frac{B}{x-\omega_2} + \frac{C}{x-\omega_3}$, where $\omega_3 = 1$. And then using Taylor expansion we easily get the closed formula for S_n.

(c) Similar to part (b) though a little more complex. Try it if you're interested!