The Power of Generating Functions

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Once my officemate Tieming asked me about a problem that she met in her research. Suppose $\boldsymbol{B}$ is a symmetric matrix of huge dimension and $\boldsymbol{D}$ is a diagonal matrix with nonnegative diagonal elements. We already know the inverse of $\boldsymbol{B}$, how can we calculate the inverse of $\boldsymbol{B}+\boldsymbol{D}$? I thought for a while and found a good way to solve the problem.

First I recalled the following results (easy to check): if $\boldsymbol{B}$ is a $k \times k$ non-singular matrix and $\boldsymbol{B}+\boldsymbol{c}\boldsymbol{c}'$ is non-singular, then

$$\left(\boldsymbol{B}+\boldsymbol{c}\boldsymbol{c}'\right)^{-1} = \boldsymbol{B}^{-1} - \frac{\boldsymbol{B}^{-1}\boldsymbol{c}\boldsymbol{c}'\boldsymbol{B}^{-1}}{1+\boldsymbol{c}'\boldsymbol{B}^{-1}\boldsymbol{c}}.$$

If the diagonal matrix $\boldsymbol{D}$ can be written as $\boldsymbol{c}\boldsymbol{c}'$ for some vector $\boldsymbol{c}$, then we know how to calculate the inverse. But unfortunately usually a diagonal matrix $\boldsymbol{D}$ cannot be written as $\boldsymbol{c}\boldsymbol{c}'$. However, suppose $\boldsymbol{D} = diag(d_1,\ldots,d_n) = \sum_{i=1}^{n} diag(d_i\boldsymbol{e}_i)$, where $\boldsymbol{e}_i$ is a vector of length n with all elements 0 except the the i^{th} element. It's obvious that $diag(d_i\boldsymbol{e}_i) \equiv \boldsymbol{D}_i$ is a diagonal matrix whose i^{th} diagonal element is d_i and all other elements are 0. $\boldsymbol{D}_i$ can be written as $\boldsymbol{c}_i\boldsymbol{c}_i'$ for some $\boldsymbol{c}_i$, so $\boldsymbol{B}+\boldsymbol{D}$=$\boldsymbol{B}+\sum_{i=1}^{n} D_i$=$\boldsymbol{B}+\sum_{i=1}^{n-1} D_i + D_n$. Thus we can easily calculate the inverse of $\boldsymbol{B}+\boldsymbol{D}$ if we know the inverse of $\boldsymbol{B}+\sum_{i=1}^{n-1} D_i$. We can easily calculate the inverse of $\boldsymbol{B}+\sum_{i=1}^{n-1} D_i$ if we know the inverse of $\boldsymbol{B}+\sum_{i=1}^{n-2} D_i$, so on and so forth. This indicates that we can calculate the inverse of $\boldsymbol{B}+\boldsymbol{D}$ iteratively.

I haven't implemented this algorithm yet, but a roughly estimate of the complexity of this algorithm tells me that even R can handle it. I'll write a R function to do this later when I'm free.